

Model independent study of massive lepton elastic scattering on the proton, beyond the Born approximation

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Abstract

Model independent expressions for all polarization observables in $\mu + p \rightarrow \mu + p$ elastic scattering are obtained, taking into account the lepton mass and including the two-photon exchange contribution. The spin structure of the matrix element is parametrized in terms of six independent complex amplitudes, functions of two independent kinematical variables. General statements about the influence of the two-photon-exchange terms on the differential cross section and on polarization observables are given. Polarization effects have been investigated for the case of a longitudinally polarized

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1. Introduction

In this work, elastic lepton nucleon scattering is revised, deriving the kinematical relations and the expressions of the observables in the most general form, taking into account the lepton mass and expressing the matrix element as function of six independent amplitudes.

Electron proton elastic scattering, which involves four spin 1/2 particles, is considered one of the simplest reaction to investigate the proton structure. Assuming that the interaction occurs through the exchange of a virtual photon, considering the spin one nature of the virtual photon, parity conservation, and the identity of the initial and final state bring the number of amplitudes from sixteen to two. These two amplitudes, in general complex, become real due to unitarity as the momentum transfer $q^2 = -Q^2$ is negative (space-like region).

The cross section is expressed as function of two form factors (FFs), electric G_E and magnetic G_M , as given in Ref. [1]. This formula assumes that the interaction occurs through the exchange of a virtual photon, which carries the transfer momentum q (one photon exchange approximation) and neglects the electron mass, which are considered good approximations at energies of the order of few GeV. Polarization observables for elastic electron proton scattering were derived in [2, 3].

Several experiments were carried out, starting from the ones for which Hofstadter was rewarded by Nobel prize in 1966, to the recent polarization measurements from the GEp collaboration at JLab [4]. The surprising result was that polarized and unpolarized experiments, although based on the same theoretical background (same formalism and same assumptions), ended up with inconsistent values of the FF ratio. Note that in the unpolarized case, one measures the cross section at fixed Q^2 ($Q^2 = -q^2 > 0$), changing the beam energy and the scattered electron angle, which gives access to G_E^2 and G_M^2 , whereas, in the second case, one measures the ratio of the longitudinal to transverse recoil proton polarization which is directly related to the ratio of G_E and G_M .

It turned out that ratio of the electric to the magnetic FF, reconstructed from unpolarized measurements, is close to unity and constant with Q^2

whereas the polarization method gives a monotonically decreasing behavior, and a value close to zero at $Q^2 \leq 9 \text{ GeV}^2$. This result, as well as an eventual zero crossing of this ratio, have given rise to a large theoretical and experimental work. It was suggested that the reaction mechanism (one-photon-exchange) is not a good approximation at large Q^2 [5]. The idea that two-photon exchange could play a role at large momentum transfer, was firstly discussed in the 70's [6, 7, 8], and more recently, for electron deuteron scattering (where a discrepancy between two sets of data was also found in measurements of the structure function $A(Q^2)$ [9]).

In presence of two photon exchange, the number of amplitudes which describe the reaction is essentially increased. Instead of two real amplitudes, function of one variable, Q^2 , one has three complex amplitudes (in the approximation of zero lepton mass), functions of two variables, for example the electron scattering angle and Q^2 . In Refs. [10, 11, 12] it was shown, that, in such conditions, one can still extract the real FFs, G_E and G_M , but for this aim, it is necessary to measure three T-odd or five T-even polarization observables, including triple spin observables which are expected to be of the order of the electromagnetic coupling constant, α . Alternatively, one can use polarized electron and positron beams, in the same kinematical conditions and apply a generalized polarization method.

Moreover, some of the polarization observables are proportional to the lepton mass. In this case, not only helicity conserving amplitudes, but also spin-flip amplitudes should be taken into account, as they are of the same order of magnitude. The number of relevant amplitudes from three becomes six. The additional amplitudes have been neglected in most of the works related to two photon exchange. However, if one looks to observables which are expected to be small, or to heavy lepton scattering, these amplitudes should be taken into account.

Muon-proton scattering brings the same information concerning the electromagnetic FFs. However, in case of low energy and large lepton mass the terms proportional to the lepton mass become important and the mass should be taken explicitly into account. The expressions of the kinematical relations and of the polarized and unpolarized observables are different from those currently used. In Ref. [13] the effect of the lepton mass was discussed for ep elastic scattering. The unpolarized cross section and the double spin asymmetry, when the lepton beam and the target are polarized, were calculated, in the one-photon approximation.

The terms related to the lepton mass influence also the extraction of

the charge radius of the proton from muon proton elastic scattering, which is related, in non relativistic approximation, to the limit at $Q^2 \rightarrow 0$ of the derivative of the electric FF. Note that the problem of the proton size is object of large interest, due to the recent experiment on muonic hydrogen by laser spectroscopy measurement of the $\nu p(2S-2P)$ transition frequency [14]. The result on the proton charge radius $r_c = 0.84184(67)$ fm obtained in this experiment is one order of magnitude more precise but smaller by five standard deviation compared to the best value previously assumed $r_c = 0.8768(69)$ fm (CODATA [15]). Previous best measurements include techniques based on hydrogen spectroscopy, which are more precise, but compatible with electron proton elastic scattering at small values of the four momentum transfer squared Q^2 . The most recent result from electron proton elastic scattering, $r_c = 0.879(5)_{\text{stat}}(4)_{\text{syst}}(2)_{\text{model}}(4)_{\text{group}}$ fm, can be found in Ref. [16]. The smallest value of Q^2 reached with good precision was of the order of 10^{-3} GeV².

In Ref. [17] the elastic scattering of a proton beam on an electron target has been considered. In this case it is necessary to take into account the lepton mass, due to the inverse kinematics (the projectile is heavier than the target). This reaction allows to access the region of very small transferred momenta, up to 10^{-6} GeV².

The fact that the proton charge radius was not measured in the process of the elastic muon-proton scattering led to the proposal *MUon proton Scattering Experiment* (MUSE) at Paul Scherrer Institut [18].

This experiment plans a simultaneous measurement of the elastic μ^-p and e^-p scattering as well as μ^+p and e^+p . It will establish the consistency or the difference of the muon-proton and electron-proton interaction with good precision in the considered kinematics [18]. Three values of the muon beam momenta which are comparable with the muon mass: 115 MeV, 153 MeV, and 210 MeV, were chosen. However, in case of low energy and large lepton mass, the terms proportional to the lepton mass become important and the mass should be taken explicitly into account in the calculation of the kinematical variables and of the experimental observables. The MUSE experiment intends to observe, if any, two-photon-exchange effects and to test their effect on the radius extraction. The relative precision of the cross section measurements will be about few tenth of a percent.

The muon-proton interaction is expected to have reduced radiative corrections compared to electrons, which gives a safer extraction of the physics information related to the proton structure. Moreover, the muon beam being

naturally polarized, when it is formed from pion decay, double spin polarization experiments can be foreseen, requiring to polarize the hydrogen target, or to measure the polarization of the outgoing proton. The expressions for the polarization observables are also affected by the lepton mass.

In this paper a complete derivation of polarized and unpolarized lepton proton scattering, including the six amplitudes (in presence of two-photon exchange) and taking into account the lepton mass, is presented.

Experimental observables are then derived in model independent way, as functions of the six amplitudes. We point out the relevant terms depending on the lepton mass and on the new amplitudes, and show in which limits the known formulas can be recovered, for an appropriate use of the approximated formulas existing in the literature.

2. Formalism

Let us consider the reaction:

$$\mu(k_1) + p(p_1) \rightarrow \mu(k_2) + p(p_2), \quad (1)$$

where the momenta of the particles are written in the parenthesis. In the laboratory system, where the present analysis is performed, the proton (muon) four momenta in the initial and final states are respectively p_1 and p_2 (k_1 and k_2) with components:

$$p_1 = (M, 0), \quad p_2 = (E_2, \vec{p}_2), \quad k_1 = (\varepsilon_1, \vec{k}_1), \quad k_2 = (\varepsilon_2, \vec{k}_2), \quad (2)$$

where M is the proton mass. The general structure of the matrix element of the reaction (1), taking into account the lepton mass and the two-photon-exchange contribution, can be written as sum of two terms:

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2, \quad (3)$$

where the first (second) term is determined by the spin non-flip (flip) amplitudes. The matrix element which is determined by the spin non-flip amplitudes can be written as

$$\mathcal{M}_1 = \frac{e^2}{(-q^2)} j_\mu J_\mu, \quad j_\mu = \bar{u}(k_2) \gamma_\mu u(k_1), \quad (4)$$

$$J_\mu = \bar{u}(p_2) \left[\tilde{G}_M(q^2, s) \gamma_\mu - \frac{1}{M} \tilde{F}_2(q^2, s) P_\mu + \frac{1}{M^2} \tilde{F}_3(q^2, s) P_\mu \hat{K} \right] u(p_1),$$

where $q = k_1 - k_2 = p_2 - p_1$, $s = (k_1 + p_1)^2$, $K = (k_1 + k_2)/2$, $P = (p_1 + p_2)/2$. The second term in (3) is determined by the spin flip amplitudes. Since these amplitudes are proportional to the lepton mass m , we single out explicitly the factor m/M and the matrix element can be written as

$$\begin{aligned} \mathcal{M}_2 = & \frac{m}{M} \frac{e^2}{Q^2} \left[\bar{u}(k_2) u(k_1) \bar{u}(p_2) \left(\tilde{F}_4(Q^2, s) + \frac{1}{M} \tilde{F}_5(Q^2, s) \hat{K} \right) u(p_1) + \right. \\ & \left. \tilde{F}_6(Q^2, s) \bar{u}(k_2) \gamma_5 u(k_1) \bar{u}(p_2) \gamma_5 u(p_1) \right]. \end{aligned} \quad (5)$$

where m is the lepton mass. The six complex amplitudes, $\tilde{G}_M(q^2, s)$ and $\tilde{F}_i(q^2, s)$, $i = 2 - 6$, which are generally the functions of two independent kinematical variables, q^2 and s , fully describe the spin structure of the matrix element for the reaction (1), for any number of changed virtual photons.

This expression holds under assumption of the parity (P)-invariance of the electromagnetic interaction. Note, however, that expressions (4) and (5) are not unique, but many equivalent representations of the $\mu^- p \rightarrow \mu^- p$ reaction matrix element may be written.

In the Born (one-photon-exchange) approximation these amplitudes reduce to:

$$\tilde{G}_M^{Born}(q^2, s) = G_M(q^2), \quad \tilde{F}_2^{Born}(q^2, s) = F_2(q^2), \quad \tilde{F}_i^{Born}(q^2, s) = 0, \quad i = 3 - 6, \quad (6)$$

where $G_M(q^2)$ and $F_2(q^2)$ are the magnetic and Pauli nucleon electromagnetic FFs, respectively, which are real functions of the variable q^2 in the space-like region of the momentum transfer squared. In the following we use the standard magnetic $G_M(q^2)$ and charge $G_E(q^2)$ nucleon form factors, which are related to the Pauli FF $F_2(q^2)$ by:

$$F_2(q^2) = \frac{1}{1 + \tau} (G_M(q^2) - G_E(q^2)), \quad \tau = -\frac{q^2}{4M^2}. \quad (7)$$

In analogy with this relation, let us introduce the linear combinations of the $\tilde{G}_{M,E}(q^2, s)$ amplitudes which reduce to the Pauli FF F_2 in the Born approximation:

$$\tilde{F}_2(q^2, s) = \frac{1}{1 + \tau} (\tilde{G}_M(q^2, s) - \tilde{G}_E(q^2, s)).$$

To separate the effects due to the Born and to the two-photon-exchange contribution as well as the terms induced by the lepton mass, let us single

out the dominant contribution and define the following decompositions of the amplitudes

$$\tilde{G}_M(q^2, s) = G_M(q^2) + \Delta G_M(q^2, s), \quad \tilde{G}_E(q^2, s) = G_E(q^2) + \Delta G_E(q^2, s). \quad (8)$$

The order of magnitude of these quantities is

$$\Delta G_M(q^2, s) \sim \Delta G_E(q^2, s) \sim \tilde{F}_i(q^2, s) (i = 3 - 6) \sim \alpha, \quad G_{M,E} \sim \alpha^0.$$

Since the terms ΔG_M , ΔG_E and \tilde{F}_i are small in comparison with the dominant ones, we neglect, in the following, the bilinear combinations of these small terms.

The modulus of the matrix element squared in such approximation can be written as

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + 2\text{Re}\mathcal{M}_1\mathcal{M}_2^*. \quad (9)$$

The differential cross section in terms of the modulus of the matrix element squared is

$$d\sigma = \frac{(2\pi)^4}{4I} |\mathcal{M}|^2 \frac{d\vec{k}_2 d\vec{p}_2}{(2\pi)^6 4\varepsilon_2 E_2} \delta^{(4)}(k_1 + p_1 - k_2 - p_2), \quad (10)$$

where $I^2 = (k_1 \cdot p_1)^2 - m^2 M^2$ and $\varepsilon_2(E_2)$ is the energy of the scattered muon (recoil proton).

Writing the matrix element in the form $\mathcal{M} = (e^2/(-q^2))\overline{\mathcal{M}}$, one can obtain following expression for the differential cross section of the reaction (1) in the laboratory system for the case when the scattered muon is detected in the final state

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4M} \frac{\vec{k}_2^2}{d|\vec{k}_1|} \frac{|\overline{\mathcal{M}}|^2}{q^4}, \quad (11)$$

where $d = (M + \varepsilon_1)|\vec{k}_2| - \varepsilon_2|\vec{k}_1|\cos\theta$, θ is the muon scattering angle (angle between the directions of the initial and final muons), $d\Omega$ is the differential solid angle of the scattered muon). The scattered muon energy is written in term of the muon scattering angle as

$$\varepsilon_2 = \frac{(\varepsilon_1 + M)(M\varepsilon_1 + m^2) + \vec{k}_1^2 \cos\theta \sqrt{M^2 - m^2 \sin^2\theta}}{(\varepsilon_1 + M)^2 - \vec{k}_1^2 \cos^2\theta}. \quad (12)$$

The differential cross section for the case when the recoil proton is detected in the final state ($d\Omega_p$ is the differential solid angle of the scattered proton) is:

$$\frac{d\sigma}{d\Omega_p} = \frac{\alpha^2}{4M} \frac{\vec{p}_2^2}{\bar{d}|\vec{k}_1|} \frac{|\bar{M}|^2}{q^4}, \quad (13)$$

where $\bar{d} = (M + \varepsilon_1)|\vec{p}_2| - E_2|\vec{k}_1|\cos\theta_p$, and θ_p is the angle between the directions of the muon beam and the recoil proton. Using the relation

$$dq^2 = -|\vec{k}_1||\vec{p}_2| \frac{1}{\pi} \frac{E_2 + M}{\varepsilon_1 + M} d\Omega_p,$$

we obtain the following expression for the differential cross section over the q^2 variable

$$\frac{d\sigma}{dq^2} = -\frac{\pi\alpha^2}{4M} \frac{|\vec{p}_2|}{\bar{d}|\vec{k}_1|^2} \frac{\varepsilon_1 + M}{E_2 + M} \frac{|\overline{\mathcal{M}}|^2}{q^4}. \quad (14)$$

The modulus of the first matrix element squared can be written as

$$|\overline{\mathcal{M}}_1|^2 = L_{\mu\nu} H_{\mu\nu}, \quad (15)$$

where the leptonic $L_{\mu\nu}$ and hadronic $H_{\mu\nu}$ tensors are defined as follows

$$L_{\mu\nu} = j_\mu j_\nu^*, \quad H_{\mu\nu} = J_\mu J_\nu^*. \quad (16)$$

If the initial and scattered muons are unpolarized, the leptonic tensor is

$$L_{\mu\nu}(0) = 2q^2 g_{\mu\nu} + 4(k_{1\mu}k_{2\nu} + k_{2\mu}k_{1\nu}). \quad (17)$$

where $g_{\mu\nu}$ is the symmetric tensor: $g_{\mu\nu} = (\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)/2$. In the case of polarized muon beam, the spin-dependent part of the leptonic tensor can be written as

$$L_{\mu\nu}(\xi) = 2im \langle \mu\nu q \xi \rangle, \quad (18)$$

where $\langle \mu\nu ab \rangle = \varepsilon_{\mu\nu\rho\sigma} a_\rho b_\sigma$ and ξ_μ is the muon polarization 4-vector.

The polarization 4-vector of a particle in the system where its momentum is \vec{p} is connected with the polarization vector $\vec{\chi}$ in its rest frame by a Lorentz boost

$$\vec{\xi} = \vec{\chi} + \frac{\vec{p} \cdot \vec{\chi} \vec{p}}{m(E + p)}, \quad s^0 = \frac{1}{m} \vec{p} \cdot \vec{\xi}.$$

The polarization four vector satisfies the conditions $\xi^2 = -1, k_1 \cdot \xi = 0$.

Let us introduce the following notation:

$$R_1 = |\overline{\mathcal{M}}_1|^2, \quad R_{int} = 2Re\overline{\mathcal{M}}_1\overline{\mathcal{M}}_2^*. \quad (19)$$

3. The differential cross section

In this section consider the elastic scattering of unpolarized muon beam by unpolarized proton target. The hadronic tensor $H_{\mu\nu}(0)$ can be written as

$$H_{\mu\nu}(0) = H_1 \tilde{g}_{\mu\nu} + H_2 P_\mu P_\nu + H_3 (P_\mu K_\nu + P_\nu K_\mu) + i H_4 (P_\mu K_\nu - P_\nu K_\mu), \quad (20)$$

with $\tilde{g}_{\mu\nu} = g_{\mu\nu} - q_\mu q_\nu / q^2$. With the hadronic current given by Eq. (4), the structure functions H_i , $i=1-4$, are expressed in terms of the amplitudes as:

$$\begin{aligned} H_1 &= 2q^2(G_M^2 + 2G_M \text{Re} \Delta G_M), \\ H_2 &= \frac{8}{1+\tau} [G_E^2 + \tau G_M^2 + 2G_E \text{Re} \Delta G_E + 2\tau G_M \text{Re} \Delta G_M + \\ &\quad 2 \frac{P \cdot K}{M^2} (\tau G_M + G_E) \text{Re} \tilde{F}_3], \\ H_3 &= -8\tau G_M \text{Re} \tilde{F}_3, \quad H_4 = -8\tau G_M \text{Im} \tilde{F}_3. \end{aligned} \quad (21)$$

The contraction of the leptonic $L_{\mu\nu}(0)$ and hadronic $H_{\mu\nu}(0)$ tensors gives

$$R_1(0) = 8m^2(H_1 + 2P \cdot K H_3) + 4q^2 H_1 + 2[(1+\tau)M^2 q^2 + 4(P \cdot K)^2] H_2. \quad (22)$$

The interference term R_{int} can be written as

$$\begin{aligned} R_{int}(0) &= 128m^2 \left\{ P \cdot K G_E \text{Re} \left[\tilde{F}_4 + \frac{1}{(1+\tau)} \frac{P \cdot K}{M^2} \tilde{F}_5 \right] + \right. \\ &\quad \left. \tau G_M \left[\frac{1}{(1+\tau)} \frac{(P \cdot K)^2}{M^2} - K^2 \right] \text{Re} \tilde{F}_5 \right\}. \end{aligned} \quad (23)$$

Therefore, the unpolarized differential cross section of the reaction (1) in the laboratory (Lab) system, taking into account the lepton mass and the terms due to the two-photon-exchange, can be written in the form

$$\frac{d\sigma_{un}}{d\Omega} = \sigma_0 D, \quad (24)$$

where σ_0 is the cross section for the scattering of lepton on a point-like spin 1/2 particle. It is a generalisation of the Mott cross section (including a recoil factor) to the case when the lepton mass is not neglected

$$\sigma_0 = 4 \frac{\alpha^2}{q^4} \frac{M}{d} \frac{\vec{k}_2^2}{|\vec{k}_1|} [\varepsilon_1^2 - M(M + 2\varepsilon_1)\tau]. \quad (25)$$

Note that in the limit $m = 0$ this expression reduces to the Mott cross section

$$\sigma_0(m = 0) = \frac{\alpha^2 \cos^2 \frac{\theta}{2}}{4\varepsilon_1^2 \sin^4 \frac{\theta}{2}} \left(1 + 2 \frac{\varepsilon_1}{M} \sin^2 \frac{\theta}{2} \right)^{-1}, \quad (26)$$

where θ is the Lab muon scattering angle.

The quantity D , which contains the information about the structure of the target and the effects of the two-photon-exchange mechanism, has the form

$$\begin{aligned} D = & (1 + \tau)^{-1} \left[G_E(G_E + 2Re\Delta G_E) + \tau G_M(G_M + 2Re\Delta G_M) + \right. \\ & 2 \left(\frac{\varepsilon_1}{M} - \tau \right) (G_E + \tau G_M) Re\tilde{F}_3 \Big] + \\ & \frac{1}{2} [\varepsilon_1^2 - M(M + 2\varepsilon_1)\tau]^{-1} \left\{ -\tau(q^2 + 2m^2)G_M(G_M + 2Re\Delta G_M) - \right. \\ & 4\tau m^2 \left(\frac{\varepsilon_1}{M} - \tau \right) G_M Re\tilde{F}_3 + \\ & 4\tau \frac{m^2}{M^2} (1 + \tau)^{-1} [\varepsilon_1^2 - m^2 - \tau(m^2 + M^2) - 2\tau M\varepsilon_1] G_M Re\tilde{F}_5 + \\ & \left. 4m^2 \left(\frac{\varepsilon_1}{M} - \tau \right) G_E Re \left[\tilde{F}_4 + (1 + \tau)^{-1} \left(\frac{\varepsilon_1}{M} - \tau \right) \tilde{F}_5 \right] \right\}. \end{aligned} \quad (27)$$

In case of one-photon exchange, taking into account the lepton mass, we recover the expressions of Ref. [13] and the differential cross section becomes:

$$\frac{d\sigma}{d(-q^2)} = \frac{\pi\alpha^2}{2M^2\vec{k}_1^2} \frac{\mathcal{D}}{q^4}, \quad (28)$$

with

$$\begin{aligned} \mathcal{D} = & q^2(q^2 + 2m^2)G_M^2(q^2) + \\ & 2 \left[q^2 M^2 + \frac{1}{1 + \tau} \left(2M\varepsilon_1 + \frac{q^2}{2} \right)^2 \right] [G_E^2(q^2) + \tau G_M^2(q^2)]. \end{aligned} \quad (29)$$

For experiments where the scattered lepton is detected, it may be useful to give the equivalent expression in terms of the solid angle of the lepton, $d\Omega = 2\pi d\cos\theta$:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 |\vec{k}_2|^3}{2M|\vec{k}_1|(M\varepsilon_1\varepsilon_2 - m^2 E_2)} \frac{\mathcal{D}}{q^4}, \quad (30)$$

where the relation between the energy and the angle of the scattered lepton θ was given in Eq. (12).

4. Polarization observables

The calculation of the polarization observables requires to define a coordinate frame. Let us specify the coordinate frame in the Laboratory (Lab) system: the z -axis is directed along the muon beam momentum \vec{k}_1 , the y -axis is directed along the vector $\vec{k}_1 \times \vec{k}_2$, and the x -axis is chosen in order to form a left handed coordinate system. Therefore the reaction plane is the xz -plane.

4.1. T -even polarization observables

In this chapter we consider the Time (T)-even polarization observables, which depend on the spin correlation $\vec{\xi} \cdot \vec{\xi}_1$ (polarized muon beam and proton target) and $\vec{\xi} \cdot \vec{\xi}_2$ (polarized muon beam and recoil proton). All these T-even polarization observables are determined by the real parts of the two-photon-exchange amplitudes.

4.1.1. Polarized beam and target

The hadronic tensor $H_{\mu\nu}(\xi_1)$ corresponding to the contribution of the non-flip spin amplitudes is

$$\begin{aligned} H_{\mu\nu}(\xi_1) = & H_5 P_\mu P_\nu + H_6 (P_\mu < K q \xi_1 \nu > + P_\nu < K q \xi_1 \mu >) + \\ & H_7 (P_\mu < p_1 p_2 \xi_1 \nu > + P_\nu < p_1 p_2 \xi_1 \mu >) + \\ & i H_8 < \mu \nu q \xi_1 > + i H_9 < \mu \nu q K > + i H_{10} < \mu \nu p_1 p_2 >, \end{aligned} \quad (31)$$

where $H_i (i = 4 - 10)$ are the structure functions and their expressions in terms of the amplitudes are

$$\begin{aligned} H_5 &= \frac{4}{M^3} (1 + \tau)^{-1} < P K q \xi_1 > (G_M - G_E) \text{Im} \tilde{F}_3, \quad H_6 = \frac{2}{M} G_M \text{Im} \tilde{F}_3, \\ H_7 &= \frac{2}{M} (1 + \tau)^{-1} \text{Im} (G_M \Delta G_E - G_E \Delta G_M), \\ H_8 &= -2 \left[M G_M G_E + (\varepsilon_1 - M \tau) G_M \text{Re} \tilde{F}_3 + \right. \\ &\quad \left. M (G_M \text{Re} \Delta G_E + G_E \text{Re} \Delta G_M) \right], \\ H_9 &= \frac{1}{M} q \cdot \xi_1 G_M \text{Re} \tilde{F}_3, \\ H_{10} &= \frac{1}{M} (1 + \tau)^{-1} q \cdot \xi_1 (G_M^2 - G_M G_E + 2 G_M \Delta G_M - \\ &\quad G_M \Delta G_E - G_E \Delta G_M). \end{aligned} \quad (32)$$

Note that symmetrical part of the tensor $H_{\mu\nu}(\xi_1)$ is determined by the two-photon-exchange amplitudes only.

Consider the scattering of the polarized muon beam on the polarized proton target. In this case we have

$$\begin{aligned}
R(\xi, \xi_1) &= 16mM(q \cdot \xi q \cdot \xi_1 - q^2 \xi \cdot \xi_1)(G_M G_E + G_M \text{Re} \Delta G_E + \\
&\quad G_E \text{Re} \Delta G_M) + 16mM \frac{\tau}{1+\tau} P \cdot \xi q \cdot \xi_1 [G_M(G_M - G_E) + \\
&\quad 2G_M \text{Re} \Delta G_M - G_M \text{Re} \Delta G_E - G_E \text{Re} \Delta G_M] + \\
&\quad 8m [\varepsilon_1 q \cdot \xi q \cdot \xi_1 - q^2(\varepsilon_1 - M\tau) \xi \cdot \xi_1] G_M \text{Re} \tilde{F}_3, \\
\\
R_{int}(\xi, \xi_1) &= 8 \frac{m}{M} G_M \{ -q^2 \xi \cdot \xi_1 (K \cdot P \text{Re} \tilde{F}_4 + K^2 \text{Re} \tilde{F}_5) + \\
&\quad (K^2 q \cdot \xi q \cdot \xi_1 + q^2 K \cdot \xi K \cdot \xi_1) \text{Re} \tilde{F}_5 + \\
&\quad [M \varepsilon_1 q \cdot \xi q \cdot \xi_1 + \frac{q^2}{2} (q \cdot \xi q \cdot \xi_1 + 2p_1 \cdot \xi K \cdot \xi_1)] \text{Re} \tilde{F}_4 \} - \\
&\quad 8 \frac{m}{M} \frac{G_M - G_E}{M^2(1+\tau)} < PKq\xi > < PKq\xi_1 > \text{Re} \tilde{F}_5 + \\
&\quad + 2 \frac{m}{M} [q^2 G_M (q^2 \xi \cdot \xi_1 - 2q \cdot \xi k_1 \cdot \xi_1) + \\
&\quad 2q \cdot \xi_1 \frac{\tau G_M + G_E}{1+\tau} (q^2 p_1 \cdot \xi - 2M \varepsilon_1 q \cdot \xi)] \text{Re} \tilde{F}_6. \tag{33}
\end{aligned}$$

The differential cross section of the reaction (1) describing the scattering of polarized muon beam on a polarized proton target can be written as (we give here only spin-dependent part of the cross section which is determined by the spin correlation coefficients, C_{ij}):

$$\frac{d\sigma(\xi, \xi_1)}{d\Omega} = \frac{d\sigma_{un}}{d\Omega} (1 + C_{xx} \xi_x \xi_{1x} + C_{yy} \xi_y \xi_{1y} + C_{zz} \xi_z \xi_{1z} + C_{xz} \xi_x \xi_{1z} + C_{zx} \xi_z \xi_{1x}), \tag{34}$$

where the vector $\vec{\xi}(\vec{\xi}_1)$ is the unit polarization vector in the rest frame of the lepton beam (proton target) and the spin correlation coefficients have the following form in terms of the amplitudes

$$\begin{aligned}
\overline{DC}_{xx} &= 2 \frac{m}{M} \left\{ G_M \left[q^2 G_E + \frac{\vec{k}_2^2 \sin^2 \theta}{1+\tau} (G_E + \tau G_M) \right] + \right. \\
&\quad \left. q^2 (G_E \text{Re} \Delta G_M + G_M \text{Re} \Delta G_E) + \right.
\end{aligned}$$

$$\begin{aligned}
& \left. \frac{\vec{k}_2^2 \sin^2 \theta}{1 + \tau} (G_E \text{Re} \Delta G_M + G_M \text{Re} \Delta G_E + 2\tau G_M \text{Re} \Delta G_M) \right\} + \\
& \frac{m}{M} G_M \left\{ \left[-\tau q^2 + \frac{\varepsilon_1}{M} (q^2 + \vec{k}_2^2 \sin^2 \theta) \right] \text{Re} (2\tilde{F}_3 + \tilde{F}_4 - \tilde{F}_6) + \right. \\
& \left. \left[\tau q^2 + \frac{m^2}{M^2} (q^2 + \vec{k}_2^2 \sin^2 \theta) \right] \text{Re} \tilde{F}_5 \right\} + \\
& \frac{m\varepsilon_1}{M^2} \left[q^2 G_M + \frac{\vec{k}_2^2 \sin^2 \theta}{1 + \tau} (G_M - G_E) \right] \text{Re} \tilde{F}_6, \tag{35}
\end{aligned}$$

$$\begin{aligned}
\overline{DC}_{yy} = & 2\frac{m}{M} \left\{ 2q^2 (G_M G_E + G_E \text{Re} \Delta G_M + G_M \text{Re} \Delta G_E) + \right. \\
& q^2 G_M \left[\left(\frac{\varepsilon_1}{M} - \tau \right) \text{Re} (2\tilde{F}_3 + \tilde{F}_4) + \left(\tau + \frac{m^2}{M^2} \right) \text{Re} \tilde{F}_5 + \tau \text{Re} \tilde{F}_6 \right] - \\
& \left. \frac{\vec{k}_1^2 \vec{k}_2^2 \sin^2 \theta}{M^2 (1 + \tau)} (G_M - G_E) \text{Re} \tilde{F}_5 \right\},
\end{aligned}$$

$$\begin{aligned}
\overline{DC}_{xz} = & -2\frac{m}{M} |\vec{k}_2| (|\vec{k}_1| - |\vec{k}_2| \cos \theta) \sin \theta \left\{ 2(1 + \tau)^{-1} [G_M (G_E + \tau G_M) + \right. \\
& 2\tau G_M \text{Re} \Delta G_M + G_E \text{Re} \Delta G_M + G_M \text{Re} \Delta G_E] + \\
& \frac{\varepsilon_1}{M} G_M \text{Re} (2\tilde{F}_3 + \tilde{F}_4) + \frac{m^2}{M^2} G_M \text{Re} \tilde{F}_5 - \\
& \left. \frac{\varepsilon_1}{M} (1 + \tau)^{-1} (G_E + \tau G_M) \text{Re} \tilde{F}_6 \right\} + \\
& 4\frac{m}{M} \tau |\vec{k}_1| |\vec{k}_2| \sin \theta G_M \text{Re} (\tilde{F}_4 + \tilde{F}_5 + \tilde{F}_6),
\end{aligned}$$

$$\begin{aligned}
\overline{DC}_{zx} = & 4\frac{|\vec{k}_2|}{M} \sin \theta \{ G_M [2\tau M |\vec{k}_1| G_M + \varepsilon_1 (|\vec{k}_2| \cos \theta - \\
& |\vec{k}_1|) (1 + \tau)^{-1} (\tau G_M + G_E)] + 4\tau M |\vec{k}_1| G_M \text{Re} \Delta G_M + \\
& \varepsilon_1 (|\vec{k}_2| \cos \theta - |\vec{k}_1|) (1 + \tau)^{-1} (G_E \text{Re} \Delta G_M + G_M \text{Re} \Delta G_E + \\
& 2\tau G_M \text{Re} \Delta G_M) \} + \\
& 2\frac{|\vec{k}_2|}{M} \sin \theta (\varepsilon_1 |\vec{k}_2| \cos \theta - \varepsilon_2 |\vec{k}_1|) G_M \text{Re} \left(\frac{m^2}{M^2} \tilde{F}_5 + 2\frac{\varepsilon_1}{M} \tilde{F}_3 \right) + \\
& 2\frac{m^2}{M^2} |\vec{k}_2| \sin \theta (|\vec{k}_2| \cos \theta - |\vec{k}_1|) \\
& \left[G_M \text{Re} \tilde{F}_4 - (1 + \tau)^{-1} (\tau G_M + G_E) \text{Re} \tilde{F}_6 \right],
\end{aligned}$$

$$\begin{aligned}
\overline{DC}_{zz} = & -8\tau[|\vec{k}_1|(|\vec{k}_1| - |\vec{k}_2| \cos \theta)G_M(G_M + 2\text{Re}\Delta G_M) + \\
& 2M\varepsilon_1(G_M G_E + G_E \text{Re}\Delta G_M + G_M \text{Re}\Delta G_E)] + \\
& 4\frac{\varepsilon_1}{M}(1 + \tau)^{-1}(|\vec{k}_1| - |\vec{k}_2| \cos \theta)^2[G_M(\tau G_M + G_E) + G_E \text{Re}\Delta G_M + \\
& G_M \text{Re}\Delta G_E + 2\tau G_M \text{Re}\Delta G_M] - 2\frac{\varepsilon_1^2}{M^2}\vec{k}_2^2 \sin^2 \theta G_M \text{Re}(2\tilde{F}_3 + \text{Re}\tilde{F}_4) - \\
& 4\tau(\tau M^2 - 2\vec{k}_1^2)G_M \text{Re}\tilde{F}_4 + 2\frac{m^2}{M^2}\{2\tau M^2(G_M \text{Re}\tilde{F}_4 - \\
& 2\tau G_E \text{Re}\tilde{F}_6) + \vec{k}_2^2 \sin^2 \theta[\frac{\tau G_M + G_E}{1 + \tau} \text{Re}\tilde{F}_6 - \frac{\varepsilon_1}{M} G_M \text{Re}\tilde{F}_5]\},
\end{aligned}$$

with $\overline{D} = [q^2(1 + \tau) + 4(\varepsilon_1 - M\tau)^2]D$.

4.2. T-odd spin observables

In this chapter we consider the T-odd polarization observables which depend on the T-odd polarization correlations $\vec{k}_1 \times \vec{k}_2 \cdot \vec{s}$ (the beam transverse asymmetry), $\vec{k}_1 \times \vec{k}_2 \cdot \vec{s}_1$ (the target normal spin asymmetry), $\vec{k}_1 \times \vec{k}_2 \cdot \vec{s}_2$ (the normal polarization of the recoil proton). All these T-odd polarization observables are determined by the imaginary part of the two-photon-exchange amplitudes.

4.2.1. Polarized muon beam

Let us consider the single spin asymmetry induced by the transverse polarization of the muon beam. From Eq. (20) one can see that the two-photon-exchange contribution leads to the antisymmetric part of the spin-independent hadronic tensor $H_{\mu\nu}(0)$. As a result, in the general case, a non-zero asymmetry is due to the polarization of the muon beam.

The expressions for the spin-dependent leptonic tensor and for the spin-independent hadronic tensor show that the single spin asymmetry is proportional to the two-photon-exchange term and suppressed by the factor (m/M) .

The measurement of this small observable has been done and showed that the asymmetry in the scattering of transversely polarized electrons on unpolarized protons is different from zero, contrary to what is expected in the Born (one-photon-exchange) approximation [19, 20].

In the case when only the muon beam is polarized, the differential cross section of the reaction (1) can be written as:

$$\frac{d\sigma(\xi)}{d\Omega} = \frac{d\sigma_{un}}{d\Omega}(1 + A_\mu \xi_y), \quad (36)$$

where A_μ is the beam asymmetry, i.e., the asymmetry due to the polarization of the muon beam. Note that this asymmetry is determined by the y -component of the muon polarization. The expression of the asymmetry in terms of the amplitudes is

$$DA_\mu = -\frac{m}{M} \frac{|\vec{k}_1||\vec{k}_2|\sin\theta}{\varepsilon_1^2 - M(M + 2\varepsilon_1)\tau} \text{Im} \left[\tau G_M \tilde{F}_3 + G_E \tilde{F}_4 + (1 + \tau)^{-1} \left(\frac{\varepsilon_1}{M} - \tau \right) (\tau G_M + G_E) \tilde{F}_5 \right]. \quad (37)$$

Let us enumerate the following properties of this observable:

- A_μ is proportional to the muon mass and it is determined by the muon spin component perpendicular to the reaction plane.
- A_μ is a T-odd observable and it vanishes in the Born approximation as it is determined by the imaginary part of the interference between the one- and two-photon exchange amplitudes. Thus, the asymmetry A_μ is determined by two real electromagnetic FFs $G_M(q^2)$, and $G_E(q^2)$ as well as by four complex two-photon exchange induced amplitudes: $\tilde{F}_3(q^2, s)$ (helicity conserving) and $\tilde{F}_i(q^2, s)$, ($i = 4-6$) (helicity non conserving). Therefore, this observable contains all amplitudes on equal footing, i.e., here the helicity flip amplitudes are not suppressed in comparison with the helicity conserving ones.
- A_μ vanishes, for $\theta = 0^0$ and 180^0 , as it is determined by the product $\vec{k}_1 \times \vec{k}_2 \cdot \vec{s}$, and in this case $\vec{k}_1 \parallel \vec{k}_2$.

4.2.2. Polarized proton target

Let us consider the single spin asymmetry due to the polarized proton target (called target normal spin asymmetry). Since the symmetrical part of the spin-dependent hadronic tensor $H_{\mu\nu}(\xi_1)$ is determined by the two-photon-exchange amplitudes, the target normal spin asymmetry is also determined by these amplitudes. The expressions for the corresponding contractions are the following

$$R_1(\xi_1) = \frac{8}{M} < PKq\xi_1 > \left\{ -q^2 G_M \text{Im} \tilde{F}_3 + 4(1 + \tau)^{-1} K \cdot P \text{Im}(G_M \Delta G_E - G_E \Delta G_M) + \left[-q^2 + 4(1 + \tau)^{-1} \frac{(K \cdot P)^2}{M^2} \right] (G_M - G_E) \text{Im} \tilde{F}_3 \right\}, \quad (38)$$

$$R_{int}(\xi_1) = 16 \frac{m^2}{M} < PKq\xi_1 > [G_M Im\tilde{F}_4 + (1 + \tau)^{-1} \frac{K \cdot P}{M^2} (G_M - G_E) Im\tilde{F}_5]. \quad (39)$$

When only the proton target is polarized, the differential cross section of the reaction (1) can be written as

$$\frac{d\sigma(\xi_1)}{d\Omega} = \frac{d\sigma_{un}}{d\Omega} (1 + A_p \xi_{1y}), \quad (40)$$

where A_p is the target asymmetry, i.e., the asymmetry due to the polarization of the proton target. Note that this asymmetry is determined by the y -component of the proton polarization vector. From Eq. (39) one can see that the contribution of the spin flip amplitudes is suppressed by the factor m^2/M^2 which is small also in the case of the muon scattering. So, neglecting this contribution, the asymmetry A_p can be written in terms of the amplitudes as

$$DA_p = -\frac{|\vec{k}_1||\vec{k}_2| \sin \theta}{\varepsilon_1^2 - M(M + 2\varepsilon_1)\tau} \left\{ \tau G_E Im\tilde{F}_3 + (1 + \tau)^{-1} \frac{K \cdot P}{M^2} \right. \\ \left. \left[G_M Im\Delta G_E - G_E Im\Delta G_M + \frac{K \cdot P}{M^2} (G_M - G_E) Im\tilde{F}_3 \right] \right\}. \quad (41)$$

One can see that

- A_p is determined by the component of the proton polarization vector perpendicular to the reaction plane, i.e., by the following product $\vec{k}_1 \times \vec{k}_2 \cdot \vec{s}_1$;
- A_p vanishes when $\vec{k}_1 \parallel \vec{k}_2$, i.e., in collinear kinematics;
- A_p vanishes in the Born approximation. It is determined by the interference of the one- and two-photon-exchange amplitudes, through the imaginary parts of all three complex two-photon-exchange helicity conserving amplitudes.

4.2.3. Polarization of the recoil proton

Let us consider the polarization of the recoil proton in the case when all the other particles are not polarized. In the Born approximation such polarization vanishes. The interference between the one- and the two-photon-exchange contributions induce a non-zero polarization of the recoil proton.

Note that only a measurement exists for such single spin observables: the recoil-deuteron vector polarization in unpolarized electron deuteron elastic scattering [21].

The spin-dependent part of the hadronic tensor $H_{\mu\nu}(\xi_2)$ which describes the case of the polarized recoil proton can be obtained from Eq. (31) by the following substitution: $\xi_1 \rightarrow \xi_2$. The structure functions describing this tensor can be obtained in the same way. In this case the symmetric part of the tensor $H_{\mu\nu}(\xi_2)$ is also determined by the interference of the one- and two-photon-exchange contributions. The contractions R_1 and R_{int} in this case can be written as

$$\begin{aligned}
R_1(\xi_2) &= 32M \langle PKq\xi_2 \rangle \left\{ \tau G_M \text{Im} \tilde{F}_3 + (1 + \tau)^{-1} \left[\frac{\varepsilon_1^2}{M^2} - \left(1 + 2\frac{\varepsilon_1}{M} \right) \right] \right. \\
&\quad (G_M - G_E) \text{Im} \tilde{F}_3 + m(1 + \tau)^{-1} \left(\frac{\varepsilon_1}{M} - \tau \right) \\
&\quad \left. (G_M \Delta G_E - G_E \Delta G_M) \right\}, \\
R_{int}(\xi_2) &= 32 \frac{m^2}{M} \langle PKq\xi_2 \rangle \left[G_M \text{Im} \tilde{F}_4 + (1 + \tau)^{-1} \left(\frac{\varepsilon_1}{M} - \tau \right) \right. \\
&\quad \left. (G_M - G_E) \text{Im} \tilde{F}_5 \right]. \tag{42}
\end{aligned}$$

The differential cross section of the reaction (1) can be written, for the case when only the recoil proton is polarized, as:

$$\frac{d\sigma(\xi_2)}{d\Omega} = \frac{d\sigma_{un}}{d\Omega} (1 + P_y \xi_{2y}), \tag{43}$$

where P_y is the y -component of the recoil-proton polarization vector and $\vec{\xi}_2$ is the unit polarization vector in the rest frame of the recoil proton. Note that this polarization is determined by the y -component of the recoil-proton polarization vector. From this expression one can see that the contribution of the spin flip amplitudes is suppressed by the factor m^2/M^2 which is small also in the case of the muon scattering. So, neglecting this contribution, the polarization P_y can be written in terms of the amplitudes as

$$\begin{aligned}
DP_y &= -\frac{|\vec{k}_1||\vec{k}_2| \sin \theta}{\varepsilon_1^2 - M(M + 2\varepsilon_1)\tau} \left\{ (1 + \tau)^{-1} \left(\frac{\varepsilon_1}{M} - \tau \right) \left[G_M \text{Im} \Delta G_E - G_E \text{Im} \Delta G_M + \right. \right. \\
&\quad \left. \left. + \left(\frac{\varepsilon_1}{M} - \tau \right) (G_M - G_E) \text{Im} \tilde{F}_3 \right] + \tau G_E \text{Im} \tilde{F}_3 \right\}. \tag{44}
\end{aligned}$$

One can see that

- P_y is determined by the component of the proton polarization vector perpendicular to the reaction plane, i.e., by the following product $\vec{k}_1 \times \vec{k}_2 \cdot \vec{\xi}_2$;
- P_y vanishes when $\vec{k}_1 || \vec{k}_2$, i.e., in collinear kinematics;
- P_y vanishes in the Born approximation. It is determined by the interference of the one- and two-photon-exchange amplitudes, through the imaginary parts of all three complex two-photon-exchange helicity conserving amplitudes.

4.2.4. Polarized beam and polarized recoil proton

In this section, we consider the polarization observables when the muon beam and the recoil proton are polarized (the polarization of the recoil proton is measured).

In this case, the hadronic tensor $H_{\mu\nu}(\xi_2)$ describing the contribution of the helicity conserving amplitudes has the following form:

$$\begin{aligned}
H_{\mu\nu}(\xi_2) = & g_1 P_\mu P_\nu + g_2 (P_\mu < K q \xi_2 \nu > + P_\nu < K q \xi_2 \mu >) + \\
& g_3 (P_\mu < p_1 p_2 \xi_2 \nu > + P_\nu < p_1 p_2 \xi_2 \mu >) + i g_4 < \mu \nu q K > + \\
& i g_5 < \mu \nu q \xi_2 > + i g_6 < \mu \nu p_1 p_2 >, \tag{45}
\end{aligned}$$

where $g_i (i = 1 - 6)$ are the structure functions. Their expressions in terms of the amplitudes are

$$\begin{aligned}
g_1 = & \frac{4}{M^3} (1 + \tau)^{-1} < P K q \xi_2 > (G_M - G_E) \text{Im} \tilde{F}_3, \quad g_2 = \frac{2}{M} G_M \text{Im} \tilde{F}_3, \tag{46} \\
g_3 = & \frac{2}{M} (1 + \tau)^{-1} \text{Im} (G_M \Delta G_E - G_E \Delta G_M), \quad g_4 = -\frac{1}{M} q \cdot \xi_2 G_M \text{Re} \tilde{F}_3, \\
g_5 = & -2M [G_M G_E + (\frac{\varepsilon_1}{M} - \tau) G_M \text{Re} \tilde{F}_3 + G_M \text{Re} \Delta G_E + G_E \text{Re} \Delta G_M], \\
g_6 = & \frac{1}{M} (1 + \tau)^{-1} q \cdot \xi_2 [(G_E - G_M)(G_M + \text{Re} \Delta G_M) + G_M \text{Re} (\Delta G_E - \Delta G_M)].
\end{aligned}$$

Note that the symmetrical part of the tensor $H_{\mu\nu}(\xi_2)$ is determined by the interference of the two- and one-photon-exchange amplitudes only.

In this case we have

$$\begin{aligned}
R(\xi, \xi_2) = & 16mM (q \cdot \xi q \cdot \xi_2 - q^2 \xi \cdot \xi_2) (G_M G_E + G_M \text{Re} \Delta G_E + G_E \text{Re} \Delta G_M) + \\
& 32mM \frac{\tau}{1 + \tau} P \cdot \xi q \cdot \xi_2 [(G_E - G_M)(G_M + \text{Re} \Delta G_M) +
\end{aligned}$$

$$G_M Re(\Delta G_E - \Delta G_M)] + 16 \frac{m}{M} [(p_1 \cdot k_1 - 2M^2 \tau) q \cdot s q \cdot \xi_2 - q^2(p_1 \cdot k_1 - M^2 \tau) s \cdot \xi_2] G_M Re \tilde{F}_3, \quad (47)$$

$$\begin{aligned} R_{int}(\xi, \xi_2) = & 8 \frac{m}{M} G_M [-q^2 \xi \cdot \xi_2 (K \cdot P Re \tilde{F}_4 + K^2 Re \tilde{F}_5 - M^2 \tau Re \tilde{F}_6) + \\ & q \cdot s q \cdot \xi_2 (K \cdot P Re \tilde{F}_4 + K^2 Re \tilde{F}_5 - p_1 \cdot k_1 Re \tilde{F}_6) + \\ & q^2 K \cdot s K \cdot \xi_2 Re(\tilde{F}_5 - \tilde{F}_6) + q^2 P \cdot s (K \cdot \xi_2 Re \tilde{F}_4 - P \cdot \xi_2 Re \tilde{F}_6)] \\ & + 8 \frac{m}{M} \frac{G_M - G_E}{1 + \tau} [P \cdot \xi_2 (q^2 P \cdot s - 2K \cdot P q \cdot s) Re \tilde{F}_6 - \\ & \frac{1}{M^2} < PK q \xi > < PK q \xi_2 > Re \tilde{F}_5]. \end{aligned} \quad (48)$$

The dependence of the differential cross section of the reaction (1) on the polarization transfer coefficients T_{ij} , ($i, j = x, y, z$), describing the scattering of polarized muon beam on an unpolarized proton target and measuring the recoil-proton polarization, can be written as

$$\frac{d\sigma(\xi, \xi_2)}{d\Omega} = \frac{d\sigma_{un}}{d\Omega} (1 + T_{xx} \xi_x \xi_{1x} + T_{yy} \xi_y \xi_{1y} + T_{zz} \xi_z \xi_{1z} + T_{xz} \xi_x \xi_{1z} + T_{zx} \xi_z \xi_{1x}), \quad (49)$$

The polarization transfer coefficients have the following form in terms of the FFs and two-photon-exchange amplitudes:

$$\begin{aligned} \bar{D}T_{xx} = & 2 \frac{m}{M} G_M [q^2 G_E + (1 + \tau)^{-1} k_2^2 \sin^2 \theta (G_E \\ & - \tau G_M)] + 2 \frac{m}{M} \left\{ q^2 (G_M Re \Delta G_E + G_E Re \Delta G_M) + \right. \\ & G_M Re \tilde{F}_3 \left(\frac{\varepsilon_1}{M} k_2^2 \sin^2 \theta - \tau q^2 - 4\tau M \varepsilon_1 \right) + \\ & (1 + \tau)^{-1} k_2^2 \sin^2 \theta \left[G_M Re(\Delta G_E - 2\tau \Delta G_M) + G_E Re \Delta G_M - \right. \\ & \left. 2\tau \left(1 + \frac{\varepsilon_1}{M} \right) G_M Re \tilde{F}_3 \right] \left. \right\} + 2 \frac{m}{M} q^2 G_M \left[\left(\frac{\varepsilon_1}{M} - \tau \right) Re \tilde{F}_4 + \right. \\ & \left. \left(\frac{m^2}{M^2} + \tau \right) Re \tilde{F}_5 - \tau Re \tilde{F}_6 \right] - 2 \frac{m}{M} \tau k_2^2 \sin^2 \theta \left\{ G_M Re(\tilde{F}_5 - 2\tilde{F}_4) + \right. \\ & \left. 2(1 + \tau)^{-1} \left[G_E Re \tilde{F}_6 + \left(\frac{m^2}{M^2} + \frac{\varepsilon_1}{M} \right) G_M Re \tilde{F}_5 - \right. \right. \\ & \left. \left. \frac{1}{2M} (\varepsilon_1 + \varepsilon_2) G_M Re \tilde{F}_6 \right] \right\}, \end{aligned}$$

$$\begin{aligned}
\bar{D}T_{yy} &= 2\frac{m}{M}\left\{q^2(G_M G_E + G_M Re\Delta G_E + G_E Re\Delta G_M) + \right. \\
&\quad q^2 G_M \left[\left(\frac{\varepsilon_1}{M} - \tau \right) (2Re\tilde{F}_3 + Re\tilde{F}_4) + \left(\frac{m^2}{M^2} + \tau \right) Re\tilde{F}_5 - \tau Re\tilde{F}_6 \right] - \\
&\quad \left. \frac{1}{M^2} k_1^2 k_2^2 \sin^2 \theta (1 + \tau)^{-1} (G_M - G_E) Re\tilde{F}_5 \right\}, \\
\bar{D}T_{xz} &= 2\frac{m}{M} (1 + \tau)^{-1} k_2 (k_1 - k_2 \cos \theta) \sin \theta \left[(\tau G_M - G_E) (G_M - Re\Delta G_M) + \right. \\
&\quad G_M (\tau Re\Delta G_M - Re\Delta G_E) - \frac{1}{M} (\varepsilon_1 - \tau \varepsilon_1 - 2\tau M) G_M Re\tilde{F}_3 \left. \right] - \\
&\quad 2\frac{m}{M} \frac{k_2}{M} \sin \theta G_M \left[\frac{1}{M} (k_1 - k_2 \cos \theta) Re(M\varepsilon_1 \tilde{F}_4 + m^2 \tilde{F}_5) - \right. \\
&\quad \left. 2\tau M k_2 \cos \theta Re(\tilde{F}_6 + \tilde{F}_4 - \tilde{F}_5) \right] + \\
&\quad 2\frac{m}{M} \frac{\varepsilon_1}{M} (1 + \tau)^{-1} k_2 (k_1 - k_2 \cos \theta) \sin \theta (G_E + \tau G_M) Re\tilde{F}_6 - \\
&\quad 2\frac{m}{M} (1 + \tau)^{-1} \frac{k_2}{M} (k_1 - k_2 \cos \theta) \sin \theta G_M \left\{ \tau (\varepsilon_1 + \varepsilon_2) Re(\tilde{F}_6 + \tilde{F}_4 - \tilde{F}_5) + \right. \\
&\quad \left. (\varepsilon_2 - \varepsilon_1) \left[\left(1 + \frac{\varepsilon_1}{M} \right) Re\tilde{F}_4 + \left(\frac{m^2}{M^2} - 1 \right) Re\tilde{F}_5 + Re\tilde{F}_6 \right] \right\}, \\
\bar{D}T_{zx} &= 2\frac{k_1 k_2}{M} (1 + \tau)^{-1} \sin \theta \{ [k_1 - k_2 \cos \theta - 2(1 + \tau)M] \tau G_M (G_M + 2Re\Delta G_M) + \\
&\quad (k_2 \cos \theta - \varepsilon_1) (G_M G_E + G_M Re\Delta G_E + G_E Re\Delta G_M) - \\
&\quad \frac{1}{M} G_M Re\tilde{F}_3 [(1 + \tau)\varepsilon_1 (\varepsilon_1 - k_2 \cos \theta) + \\
&\quad 2\tau (M\tau k_1 + M k_2 \cos \theta - k_1 \varepsilon_1)] \} + \\
&\quad 4\tau (1 + \tau)^{-1} m^2 \frac{k_2 \varepsilon_1 + \varepsilon_2}{k_1 M} G_M \sin \theta \left[\left(\tau - \frac{\varepsilon_1}{M} \right) Re\tilde{F}_5 - \tau Re\tilde{F}_6 - (1 + \tau) Re\tilde{F}_4 \right] - \\
&\quad 4\tau m^2 \frac{k_2}{k_1} \sin \theta \left[\frac{\varepsilon_2}{M} G_M Re(\tilde{F}_5 - \tilde{F}_4 - \tilde{F}_6) + G_M Re \left(\tilde{F}_4 - \tilde{F}_6 - \frac{m^2}{M^2} \tilde{F}_5 \right) \right. \\
&\quad \left. + \left(1 + \frac{\varepsilon_1}{M} \right) (1 + \tau)^{-1} (G_E - G_M) Re\tilde{F}_6 \right], \\
\bar{D}T_{zz} &= 2(1 + \tau)^{-1} \left\{ 2\tau^2 (M + \varepsilon_1) [2M(1 + \tau) + k_2 \cos \theta - k_1] G_M \right. \\
&\quad (G_M + 2Re\Delta G_M) + \frac{1}{M} [2M\tau (M + \varepsilon_1) (\varepsilon_1 - k_1) - k_1 k_2^2 \sin^2 \theta] \\
&\quad \left. (G_M G_E + G_M Re\Delta G_E + G_E Re\Delta G_M) \right\} +
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{M^2}(1+\tau)^{-1}G_M Re\tilde{F}_3\{k_1^2k_2^2(\tau\varepsilon_2 - \varepsilon_1)\sin^2\theta + \\
& 2M\tau(1+\tau)[\varepsilon_1^2(M+\varepsilon_1-k_1) - Mk_1(1+2\tau)(\varepsilon_1-2\tau M)]\} - \\
& 8m^2\tau^2\frac{(M+\varepsilon_1)^2}{k_1^2}\frac{\tau G_M + G_E}{1+\tau}Re\tilde{F}_6 + \\
& 8\tau^2\frac{m^2}{k_1^2}G_M Re(\tilde{F}_6 + \tilde{F}_4 - \tilde{F}_5)\left[\frac{k_1^2k_2^2}{2M^2(1+\tau)}\sin^2\theta + \right. \\
& \left. (1+2\tau)(M\varepsilon_1 + m^2) - \varepsilon_2(\varepsilon_1 + M)\right] + \\
& 8\tau\frac{m^2}{M}G_M Re\tilde{F}_5\left[\varepsilon_2 - \frac{\varepsilon_1k_2^2}{2M^2(1+\tau)}\sin^2\theta + \right. \\
& \left. \tau(M\varepsilon_1 + m^2)\frac{M+\varepsilon_1}{k_1^2}\right] - \\
& 8\tau G_M Re\tilde{F}_4\left\{m^2\left[1 - \tau\frac{(M+\varepsilon_1)^2}{k_1^2}\right] - \frac{\varepsilon_1^2k_2^2}{2M^2(1+\tau)}\sin^2\theta + \right. \\
& \left. 2\tau\left[\varepsilon_1^2 - \tau\frac{(M+\varepsilon_1)^2}{1+\tau}\right]\right\}. \tag{50}
\end{aligned}$$

5. Numerical applications

In this section we illustrate the results through numerical applications. The explicit consideration of the lepton mass modifies the kinematical variables as well as the observables. The reaction under consideration, being a binary process, two variables define completely the kinematics. Therefore, the results are preferentially illustrated as bi-dimensional plots as function of the muon beam energy and the muon scattering angle.

The effect of the muon mass is already visible on the momentum transfer squared. The relative difference of the momentum transfer squared q^2 taking and not taking into account the lepton mass is shown in Fig. 1 as a bi-dimensional plot as function of the beam energy ϵ_1 and the muon scattering angle θ , in the relevant kinematical domain. The momentum transfer squared

$$q^2 = 2m^2 - 2(\epsilon_1\epsilon_2 - k_1k_2\cos\theta)$$

is larger for electron than for a muon at the same incident energy and scattering angle. This difference is comparatively larger at small beam energies.

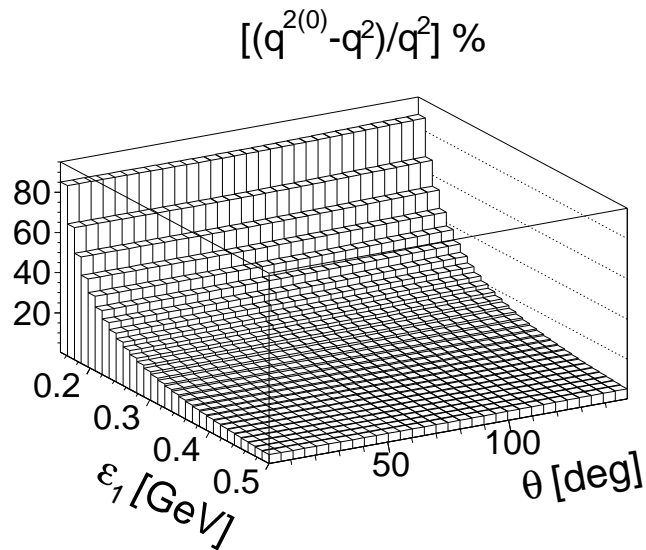


Figure 1: Bi-dimensional plot of the relative difference of momentum transfer squared as function of the incident energy and of the muon scattering angle.

In order to calculate the unpolarized and polarized observables, the amplitudes have to be calculated according to a model.

In the one-photon exchange approximation, the amplitudes reduce to the two electric G_E and magnetic G_M FFs, which we parametrize for simplicity according to a dipole dependence:

$$G_E(Q^2) = G_M(Q^2)/\mu_p = [1 + Q^2/0.71]^{-2}, \quad (51)$$

where μ_p is the anomalous magnetic moment of the proton and Q^2 is expressed in $[\text{GeV}^2]$ units. This parametrization is reasonable in the low Q^2 range considered here.

In the presence of two-photon exchange, the amplitudes are model dependent. The outcome of models can not be validated on experimental data as no clear evidence of two-photon exchange has emerged from the data up to now. In order not to mislead the reader, we will illustrate the effect of the

mass on the observables limiting the calculation to the one-photon exchange approximation. We stress, however that the complete expressions derived in this work are model independent and applicable to any model chosen for the amplitudes.

The ratio between the cross section taking and not taking into account the lepton mass is shown as function of ε_1 and θ is shown in Fig 2. One can see that this ratio increases essentially at small energies and large angles. This quantity is larger for muons than for electrons, particularly at low energies. As for electrons, the Born cross section diverges for small values of the transferred momentum, i.e., at small incident energy and/or small scattering angles.

The spin correlation coefficients when the initial particles are polarized (from Eqs. (35)) are illustrated in Fig. 2. They are sizable in particular at large angles, and show a characteristic behavior with energy and angle. The coefficients C_{xx} , C_{yy} , and C_{xz} are proportional to the lepton mass. The lepton mass for C_{zx} , and C_{zz} enters explicitly at the level of the two photon amplitudes, enhancing their effect for massive leptons.

The spin transfer coefficients, when the beam is polarized and the polarization of the recoil proton is measured, (from Eqs. (50)), are illustrated in Fig. 3.

The coefficients T_{xx} , T_{yy} , T_{xz} are proportional to the lepton mass. The mass here also plays a large role in the terms corresponding to the two-photon amplitudes. As for the spin correlations, the spin transfer coefficients show a characteristic behavior, as function of the considered kinematical variables.

6. Conclusions

The lepton hadron elastic interaction has revisited. The cross section and various polarization observables have been calculated taking into account the lepton mass, and the possible presence of two photon exchange.

The matrix element has been parametrized in the most general form, by six complex amplitudes. Model independent expressions of the observables have been given as functions of these amplitudes.

These expressions are directly applicable to low energy muon proton scattering experiments. Numerical applications have been done in the energy domain covered by planned experiments, in the one photon approximation, for dipole parametrization of the electromagnetic FFs.

The spin correlation coefficients (when the muon beam and the proton target are polarized) and the spin transfer coefficients (when the muon beam is polarized and the polarization of the recoil proton is measured) have been calculated in the relevant domain of beam energy and muon scattering angle.

It is shown that, in general, polarization observables are sizable and manifest a characteristic angular and energy dependence.

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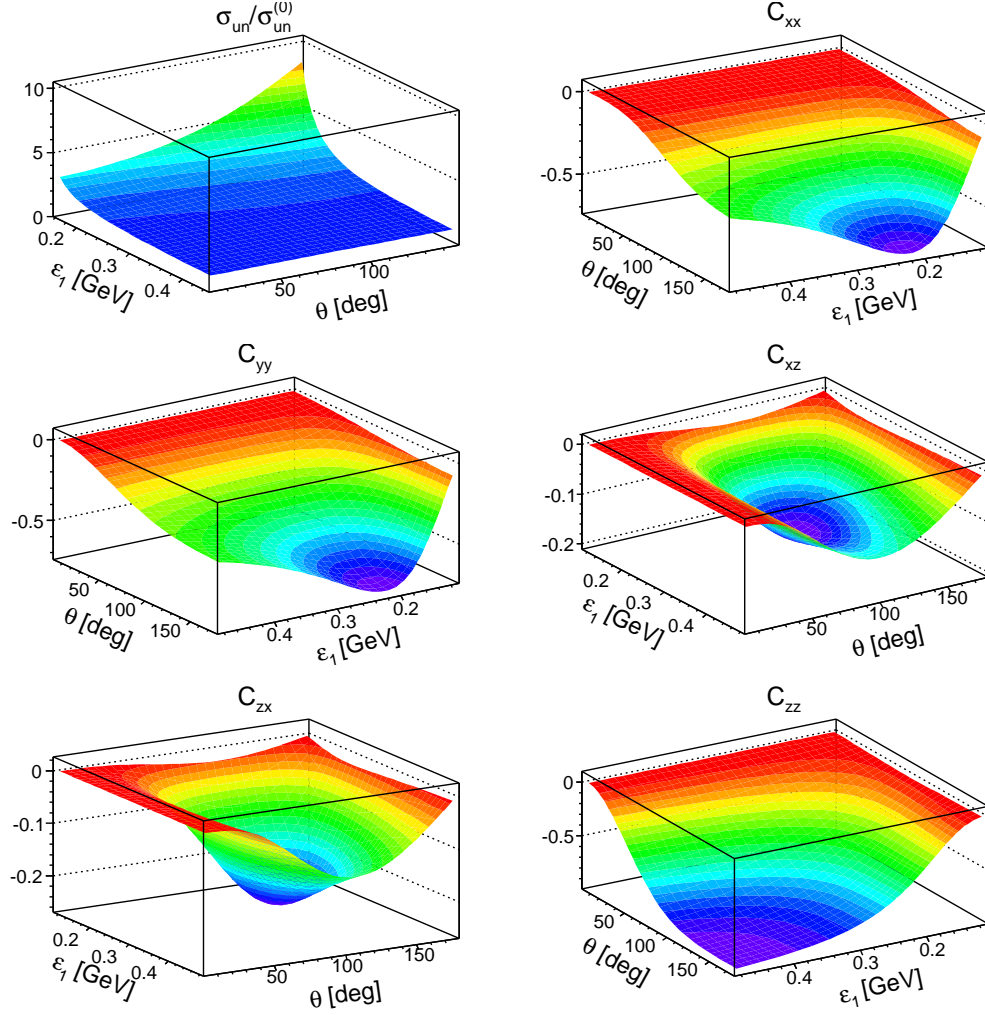


Figure 2: Observables as functions of the beam energy and of the lepton scattering angle. From left to right, from top to bottom: ratio of the muon over electron unpolarized elastic scattering cross section and spin correlation coefficients: C_{xx} , C_{yy} , C_{xz} , C_{zx} , and C_{zz} .

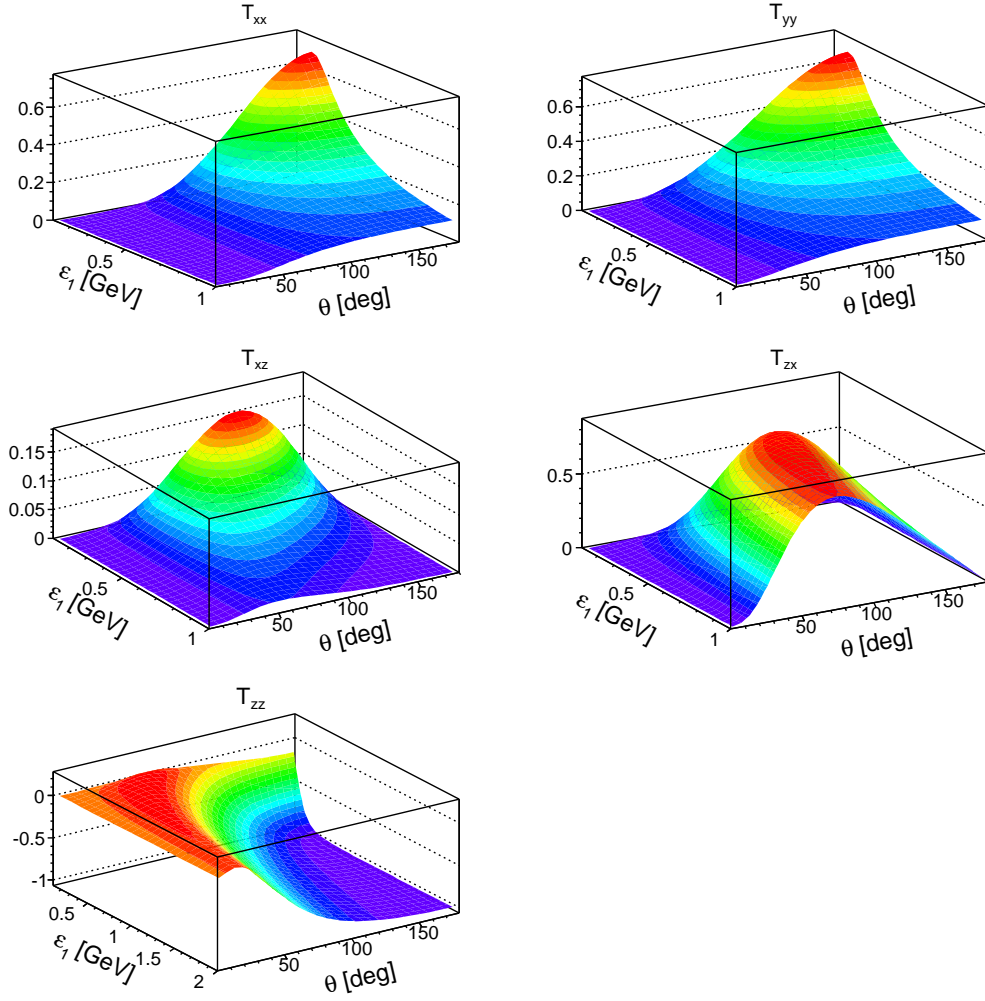


Figure 3: Spin transfer coefficients as functions of the beam energy and of the lepton scattering angle. From left to right, from top to bottom: T_{xx} , T_{yy} , T_{xz} , T_{zx} , and T_{zz} .